I Heart Cardioids

Make a Cardioid Flip Book

David Richeson

Roll a circle around another circle of the same radius. A marked point on the first circle traces a curve called a *cardioid*. (In figure 1 we rolled the orange circle around the red circle to draw the green cardioid.) This beautiful heart-shaped curve shows up in some of the most unexpected places. Grab a cup of coffee and we'll show you some.

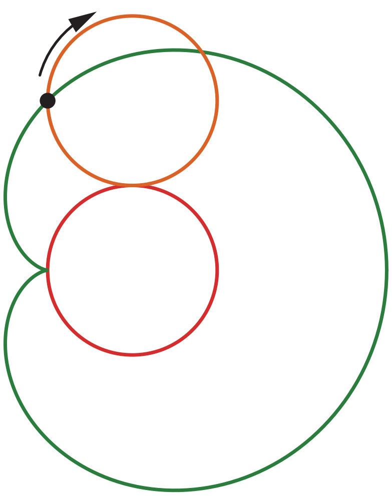

Figure 1. Roll a circle around another circle of the same radius and a point on the first circle traces a cardioid.

We do not know who discovered the cardioid. In 1637 Étienne Pascal—Blaise's father—introduced the relative of the cardioid, the limaçon, but not the cardioid itself. Seven decades later, in 1708, Philippe de la Hire computed the length of the cardioid—so perhaps he discovered it. In 1741, Johann Castillon gave the cardioid its name.

Got your coffee? Turn on the flashlight feature of your phone and shine the light into the cup from the side. The light reflects off the sides of the cup and forms a caustic on the surface of the coffee (see figure 2). This caustic is a cardioid.

The Mandelbrot set is one of the most beautiful images in all of mathematics (see figure 3). It is the set of complex numbers c such that the number 0 does not diverge to infinity under repeated iterations of the function $f_c(z) = z^2 + c$. The Mandelbrot set consists of a heart-shaped region with infinitely many circles, spiny antennae, and other heart-shaped regions growing off of it. That main heart-shaped region? It's a cardioid.

Figure 2. Shine a light from the edge of a coffee cup and the caustic it forms is a cardioid.

Cardioids even show up in audio engineering. Sometimes engineers need a uni-directional microphone—one that is very sensitive to sounds directly in front of the microphone and less sensitive to sounds next to or behind it. When they do, they reach for a *cardioid microphone*. The microphone is so-named because the graph of the sensitivity of the microphone in polar coordinates is a cardioid.

In this article, we present a few favorite places that cardioids appear. In particular, we will look at how we can use lines to construct the curved cardioid. At the end of the article, we provide a template that you can use to make your own cardioid. And we provided printable pages that can be used to make a cardioid flip book.

The Envelope of a Family of Curves

A common kids math doodle is to draw a set of coordinate axes and then draw line segments from $(0, 10)$ to $(1, 0)$, from $(0, 9)$ to $(2, 0)$, and so on, as in figure 4. This procedure magically produces a suite of lines that, when viewed together, has what appears to be a curved boundary. This curve is called the *envelope* of the family

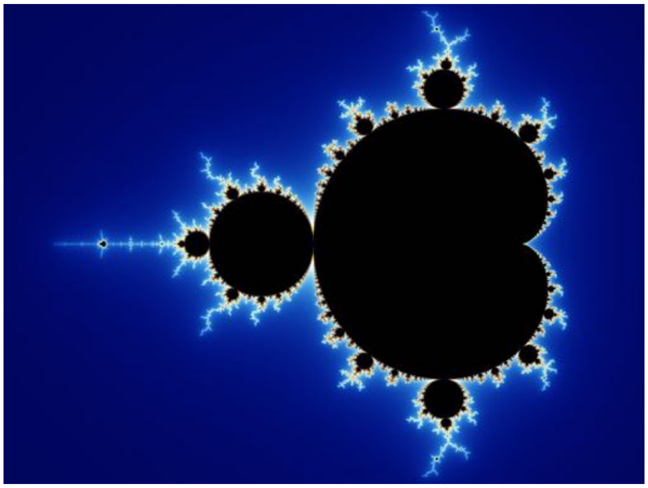

Figure 3. The main bulb of the Mandelbrot set is a cardioid.

of lines.

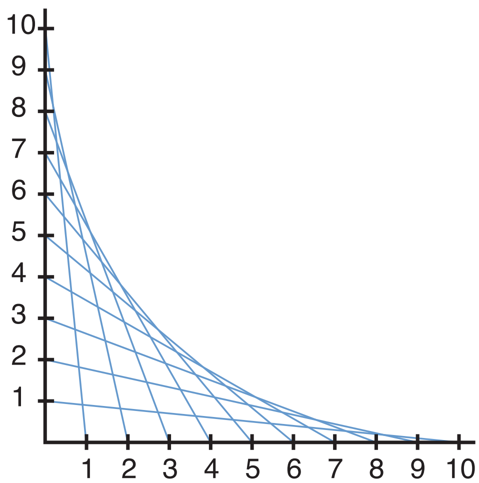

Figure 4. A curve as an envelope of lines.

Let C_t denote a family of curves parametrized by t. We can represent them as $F(x, y, t) = 0$ for some function $F : \mathbb{R}^3 \to \mathbb{R}$. For instance, in this elementary example, the line C_t joins $(0, 11-t)$ to $(t, 0)$, so it corresponds to $F(x, y, t) = yt + (11 - t)(x - t) = 0$.

Let us look at some features of this envelope. First, each line C_t is tangent to the curve. Second, if we take two nearby lines C_t and C_{t+h}, their point of intersection is near the curve, and taking the limit as $h \to 0$ yields a point on the curve. We could use either of these observations to produce a definition of an envelope, but instead, we use calculus.

In the following definition we let $F_t = \frac{\partial F}{\partial t}$ denote the partial derivative of F with respect to t.

Definition. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. The *envelope* of the set of curves $F(x, y, t) = 0$ is the set of points (x, y) such that both $F(x, y, t) = 0$ and $F_t(x, y, t) = 0$ for some value of t.

This is a mysterious definition. Why does it produce the envelope? For a fixed t and any $h \approx 0$, the curves $F(x, y, t) = 0$ and $F(x, y, t + h) = 0$ (that is, C_t and C_{t+h}) cross at a point near the envelope. Solving this pair of equations for x and y is equivalent to solving $F(x, y, t) = 0$ and $\frac{1}{h}(F(x, y, t + h) - F(x, y, t))$ for x and y. Then, as $h \to 0$, the point of intersection approaches a point on the curve. Thus, we find the point by solving $F(x, y, t) = 0$ and

$$\lim_{h\to 0} \frac{F(x, y, t + h) - F(x, y, t)}{h} = F_t(x, y, t) = 0$$

for x and y.

Returning to our example in figure 4, $F_t(x, y, t) = y - x - 11 + 2t$. If we set this expression equal to 0, solve for t, and substitute it into $F(x, y, t) = 0$, we obtain the equation $(x + y - 11)^2 - 4xy = 0$, which is a parabola opening along the line $y = x$. We can see this curve more clearly if we extend our figure beyond 1 through 10 (see figure 5).

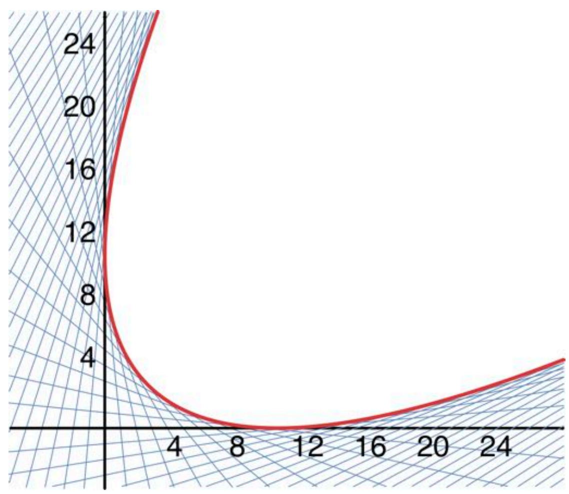

Figure 5. The envelope of lines is a parabola.

A Cardioid as an Envelope of Lines

It turns out that we can construct the cardioid as the envelope of curves, and we can do so in a number of different ways. For instance, pick a point P on a circle (the blue circle in figure 6, say). Draw circles with centers on the original circle that pass through P. The envelope of these circles is a cardioid.

But we will focus on a different example. Begin with a circle (the red circle in figure 7). Mark a certain number of evenly spaced points around the circle, N, say, and number them consecutively starting at some point P: $0, 1, 2, \ldots, N - 1$. Then for each n, draw a line between points n and $2n$ (mod N). In our example, $N = 54$, so we would join points 5 and 10, 19 and 38, and 31 and 8

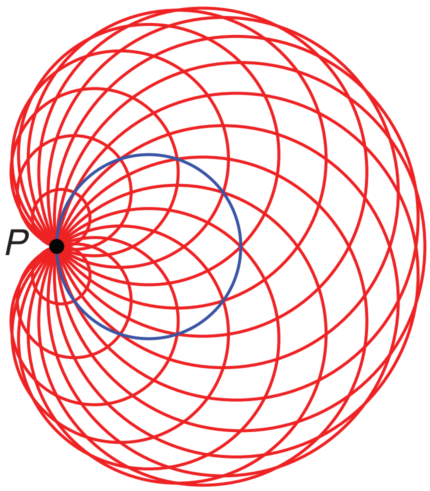

Figure 6. A cardioid as an envelope of circles.

(since 8 is 62 mod 54). The envelope of these lines is a cardioid.

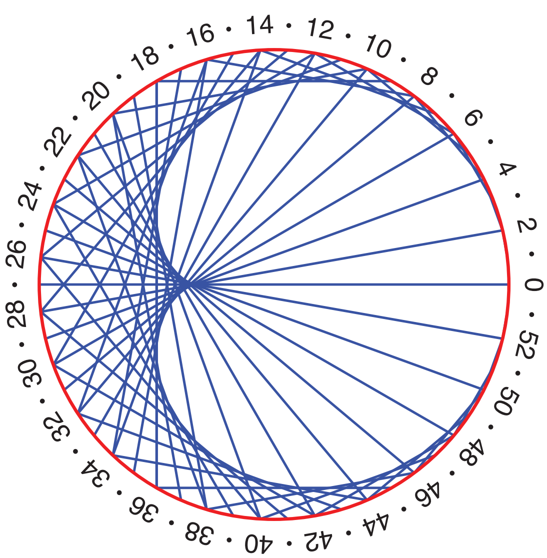

Figure 7. A cardioid as an envelope of lines.

Let's see why this is the case. Suppose our circle has center $(1, 0)$ and radius 3 and that $P = (4, 0)$. Now, starting at P, find points t and $2t$ radians around the circle from P, and draw the line segment joining them. We will show show that the envelope of all such lines is the cardioid with polar equation $r = 2(1 + \cos\theta)$.

The two points on the circle—corresponding to t and $2t$—have coordinates $(3\cos t + 1, 3\sin t)$ and $(3\cos(2t) + 1, 3\sin(2t))$. The line joining them is

$$y - 3\sin t = \left(\frac{\sin(2t) - \sin t}{\cos(2t) - \cos t}\right)(x - 3\cos t - 1).$$

After some some algebra and some applications of double angle formulas, we can express this line as

$$(\cos(2t) - \cos t)y - (\sin(2t) - \sin t)x + \sin(2t) + 2\sin t = 0.$$

In particular, the expression on the left is our function $F(x, y, t)$. Taking the partial derivative of F with respect to t we obtain

$$\begin{aligned} F_t(x, y, t) =& (-2\sin(2t) + \sin t)y - (2\cos(2t) + \cos t)x \\ &+ 2\cos(2t) + 2\cos t. \end{aligned}$$

Now, we want to show that the x and y coordinates at which $F(x, y, t) = F_t(x, y, t) = 0$ is a point on the cardioid $r = 2(1 + \cos\theta)$. The cardioid has one more surprise for us: This happens when $t = \theta$ (see figure 8)! We can express this polar curve with parametric equations as

$$\begin{aligned} x &= 2(1 + \cos\theta)\cos\theta \\ y &= 2(1 + \cos\theta)\sin\theta. \end{aligned}$$

And when we replace θ with t and substitute these expressions for x and y in F and F_t, we obtain 0. (The tedious calculations require both algebra and further applications of the double angle formula.) Thus, the cardioid is the envelope of this family of lines.

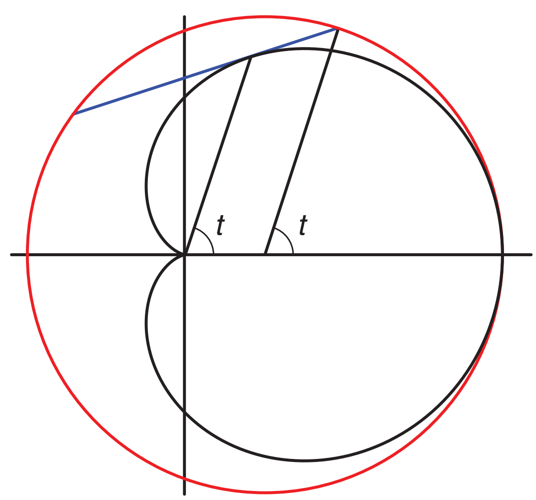

Figure 8. The secant line joining points on the circle with central angles t and $2t$ meets the cardioid at the point with polar angle t.

Back to the Coffee Cup

It turns out that this analysis explains the cardioid in the coffee cup. We can view the caustic as an envelope of lines. As we see in figure 9, if we draw lines emanating from a single point P on the circle and allow them to reflect off the circle (the angle of incidence equalling the angle of reflection), then the cardioid is the envelope of these lines.

If the light source is located at point P, then a beam of

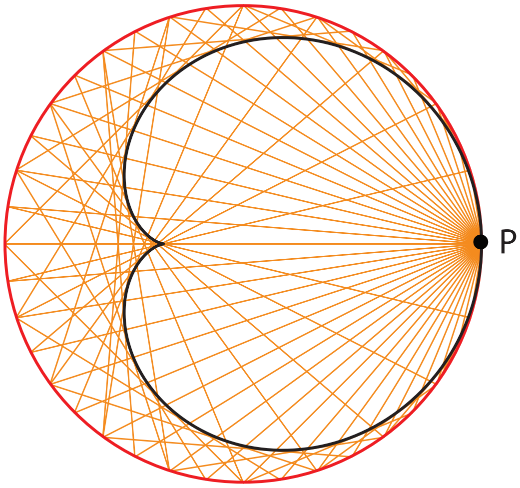

Figure 9. Draw segments from a point on a circle and have them reflect off the second point. The envelope of these lines is a cardioid.

light will reflect off a point Q on the circle and strike the circle again at R (see figure 10). Since arc PQ equals arc QR, arc PR is twice arc PQ. But then segment QR is a line that we would have drawn in the previous construction.

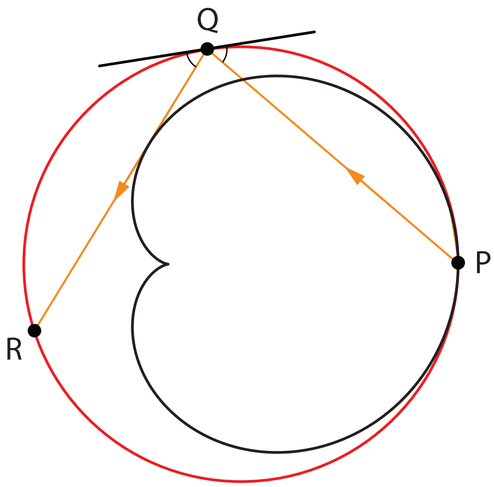

Figure 10. If a ray of light emanating from *P* strikes the circle at *Q* and then *R*, then arc *PR* is twice arc *QR*.

The coffee cup example requires one final comment. In reality, the light source will probably not be at the edge of the coffee cup, but rather, it will be far away from the cup. In this case, the rays of light are roughly parallel when they reach the cup. In this case, the curve won't be a cardioid, but its cousin—a *nephroid*. This is the envelope of lines one obtains by joining n and $3n$. In particular, as we see in figure 11, arc QR is twice arc PQ. (So in our numbering, $n = 0$ sits at the point P.)

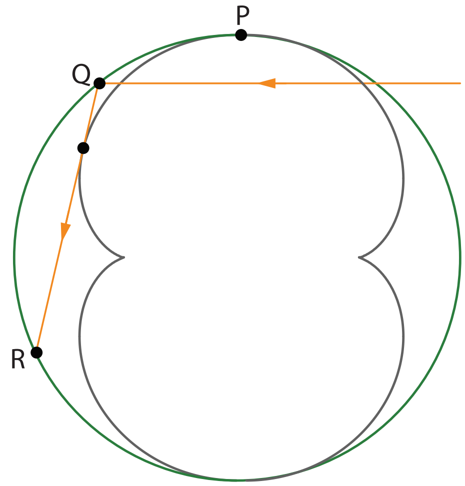

Figure 11. When the light rays come in parallel to one another, the caustic is a nephroid.

Draw Your Own Cardioid

The following page has a circle with 60 numbered points. Connect each number n to the number $2n$ mod 60 to obtain a cardioid. For a little extra fun, try connecting n to $3n$ or $4n$ or $5n$ to see what shapes you obtain.

Flip Book Instructions

The final 12 pages of the article are a printable flip book. Print the pages double-sided. The pages are designed so that the mathematical figure is on one side and the flip book page number is on the reverse side. Cut out each page, put in numerical order, and secure with a binder clip. Flip through the pages and see the animation in action!

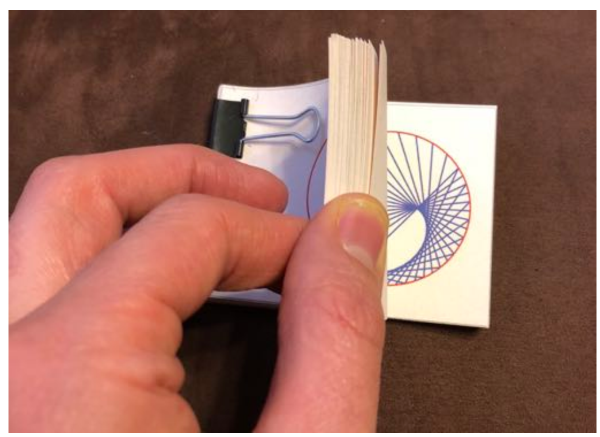

Figure 12. Flip book.

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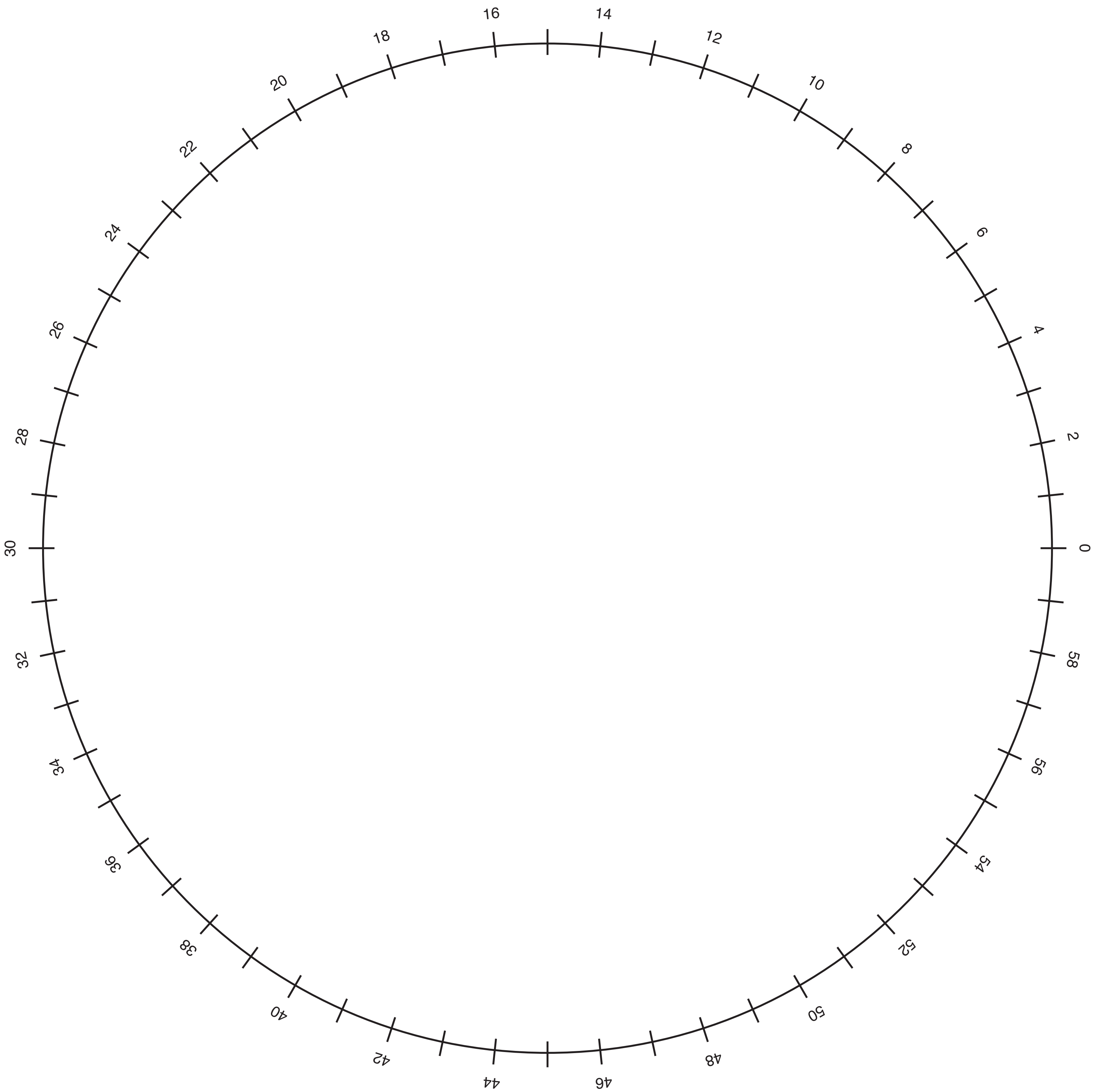
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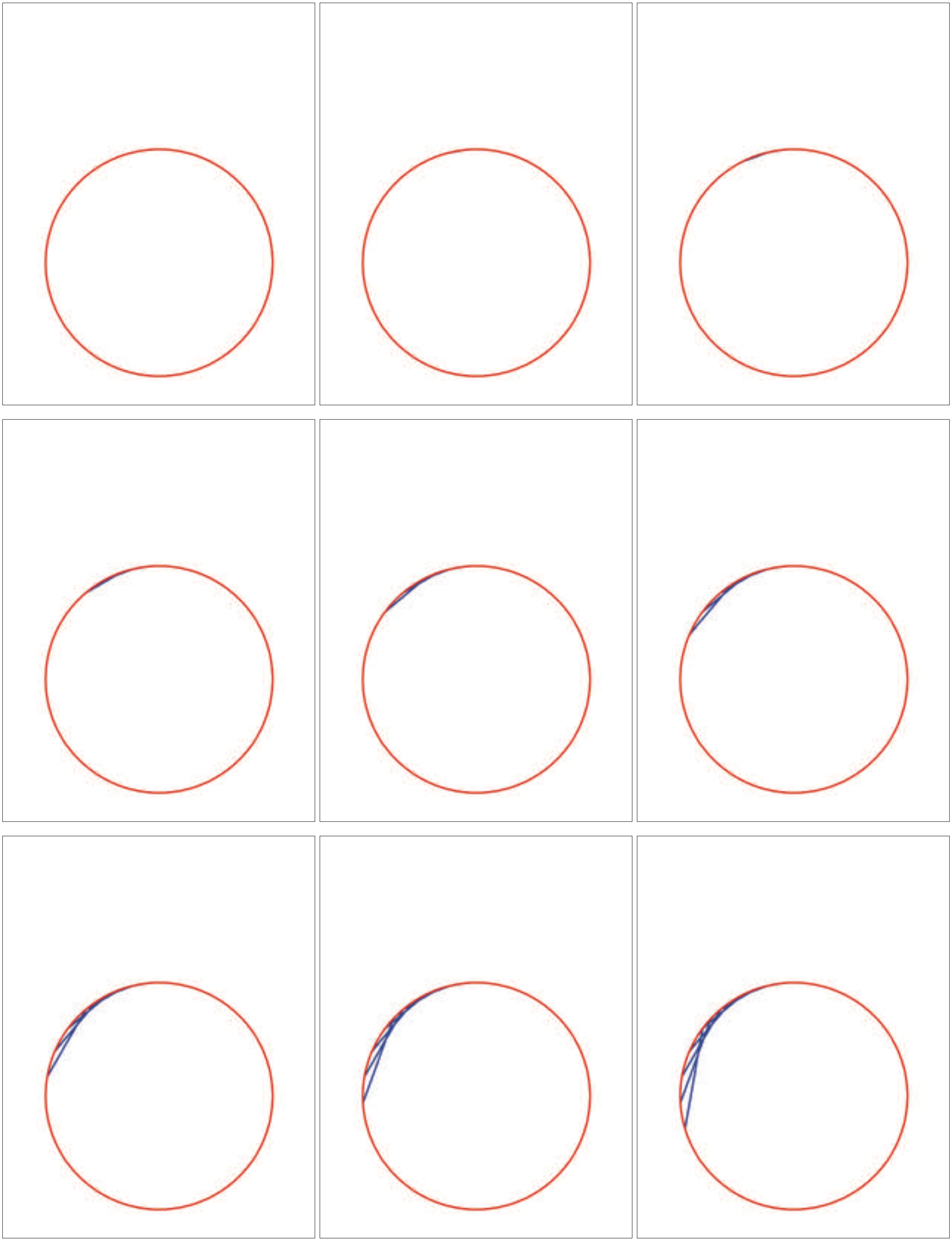

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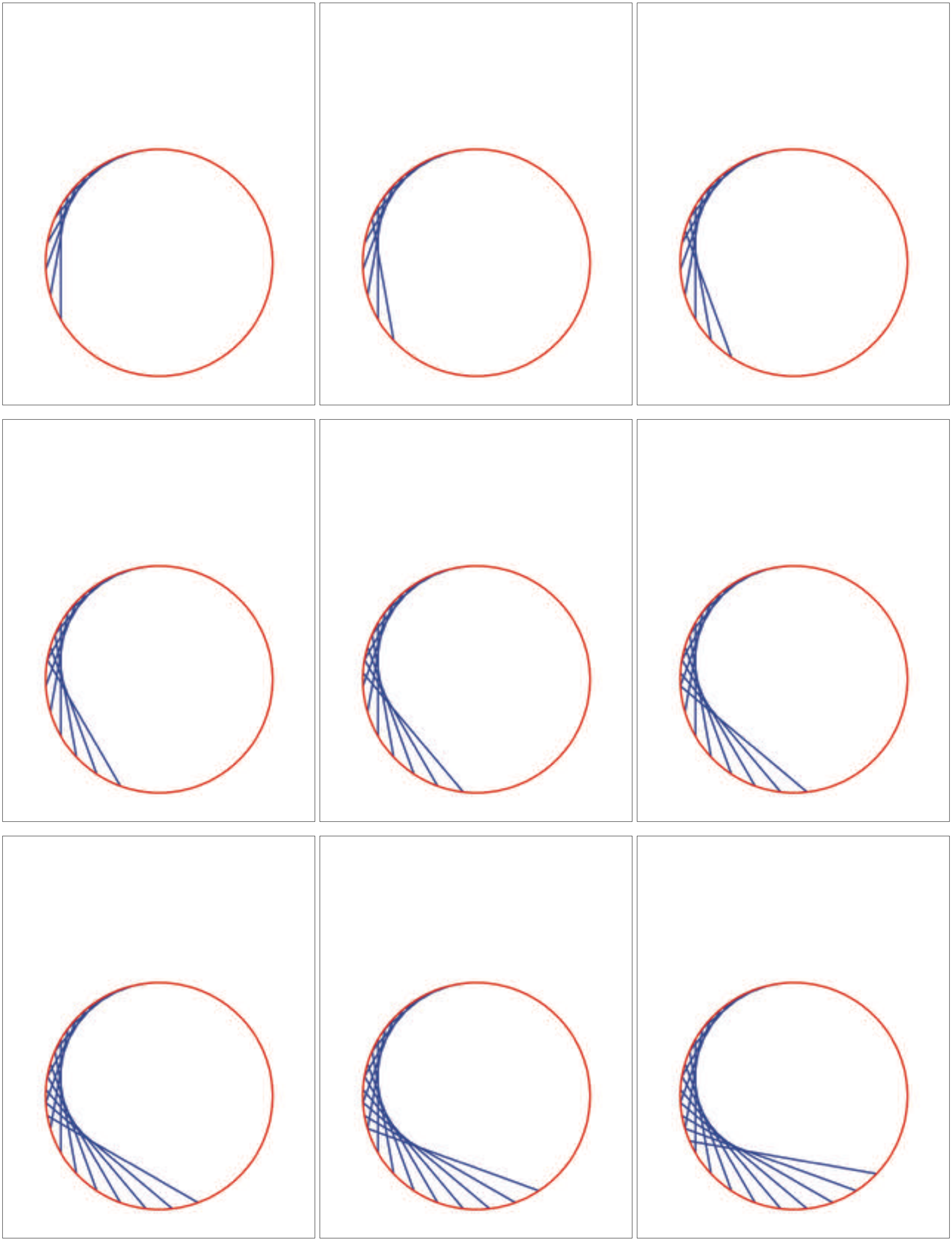

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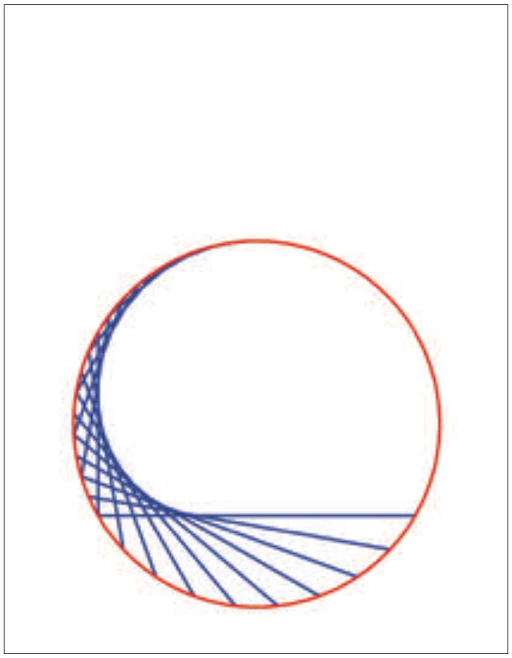
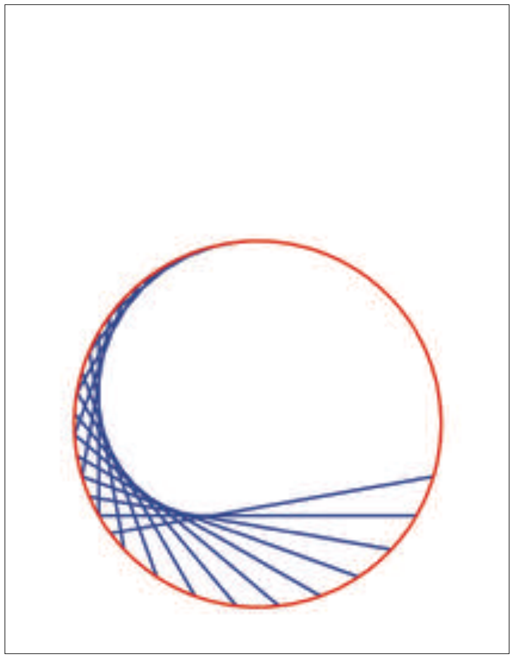
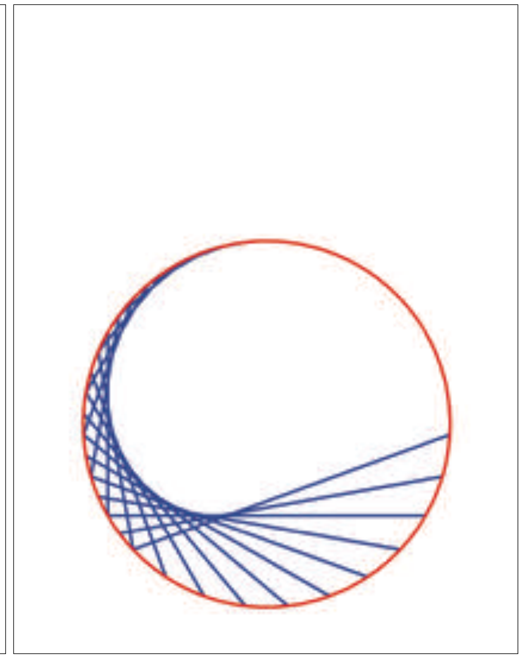
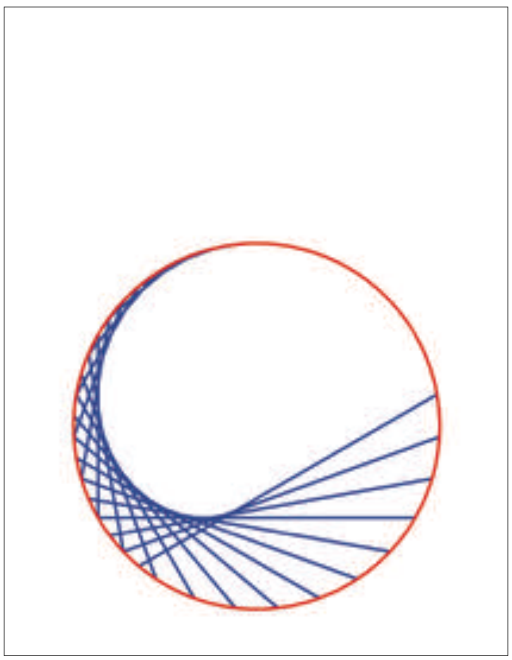
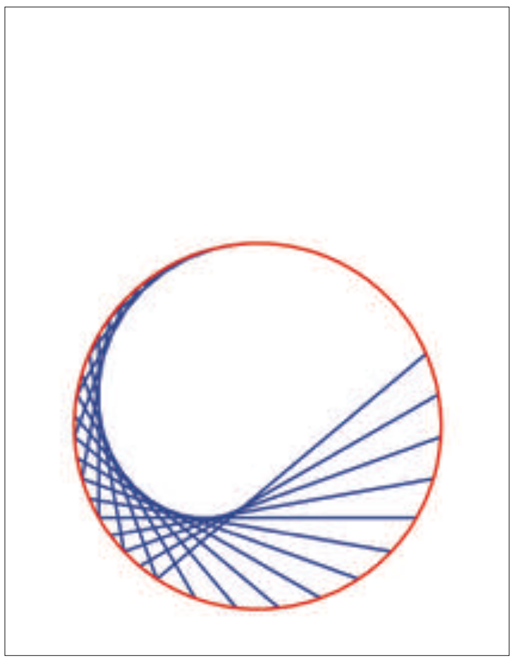
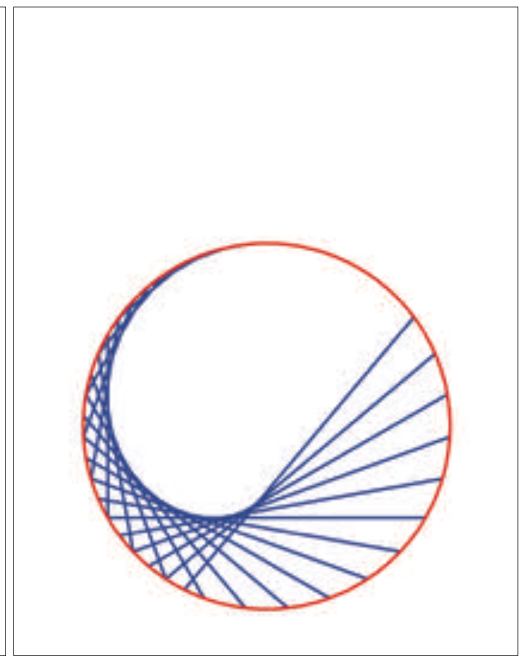
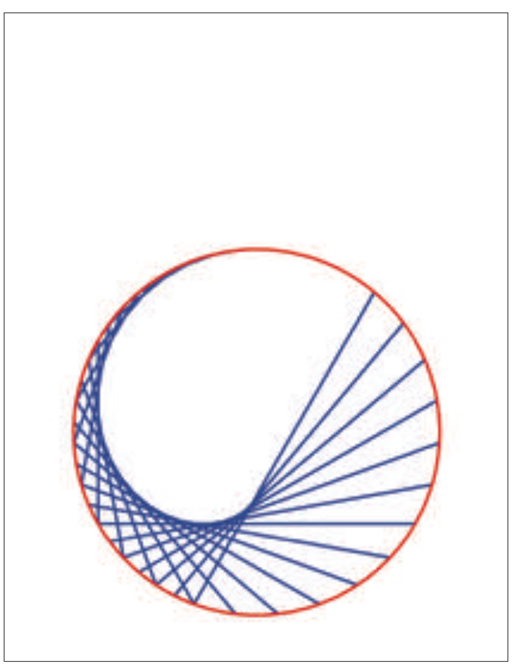
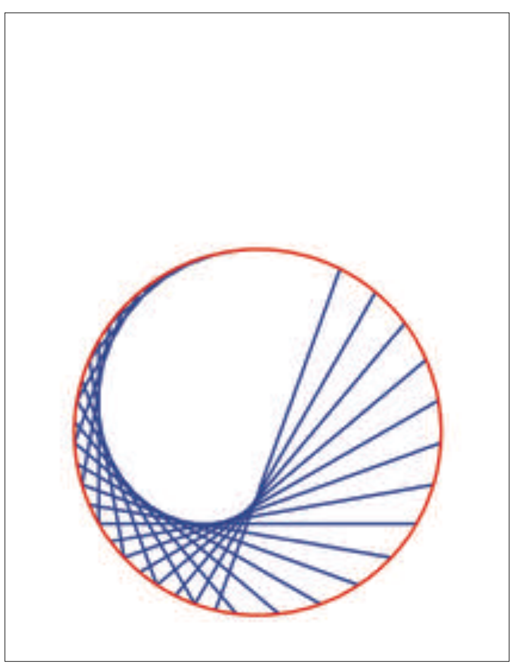
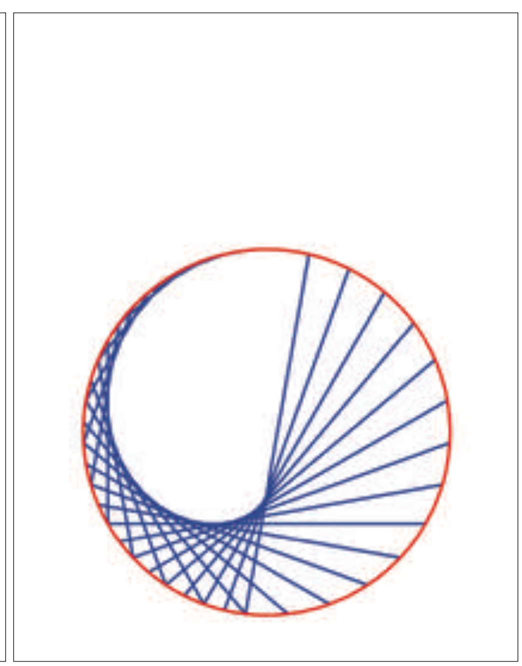

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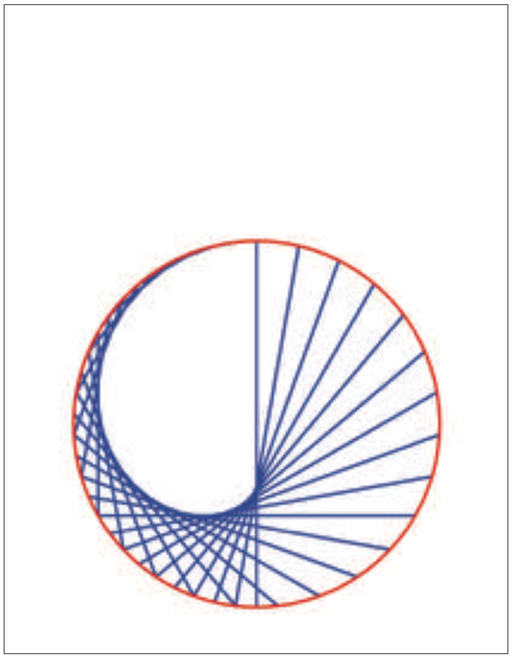
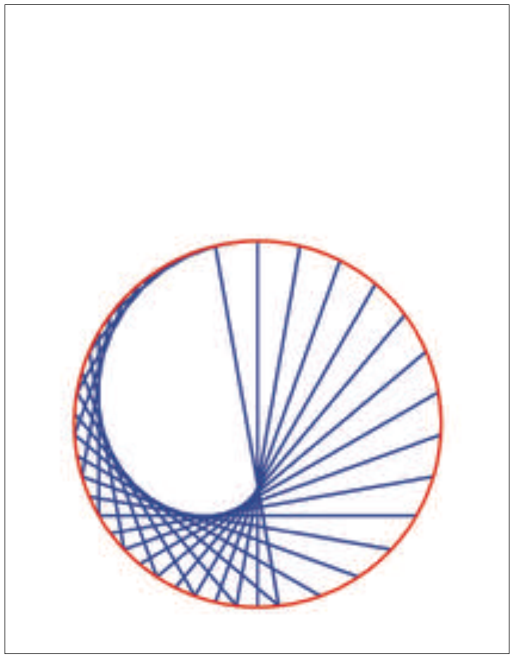
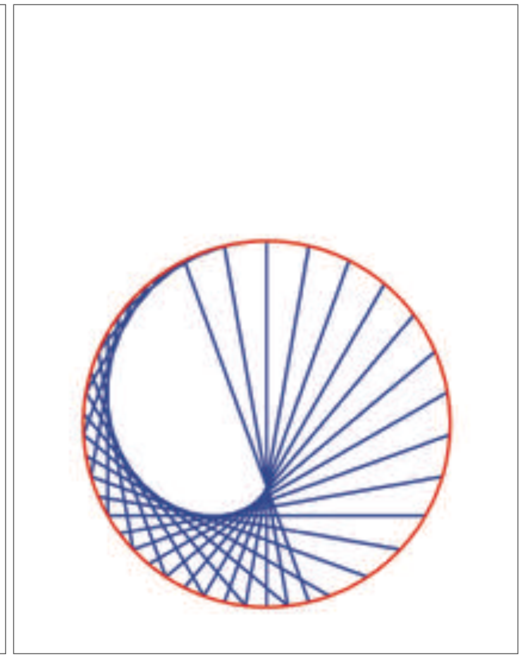
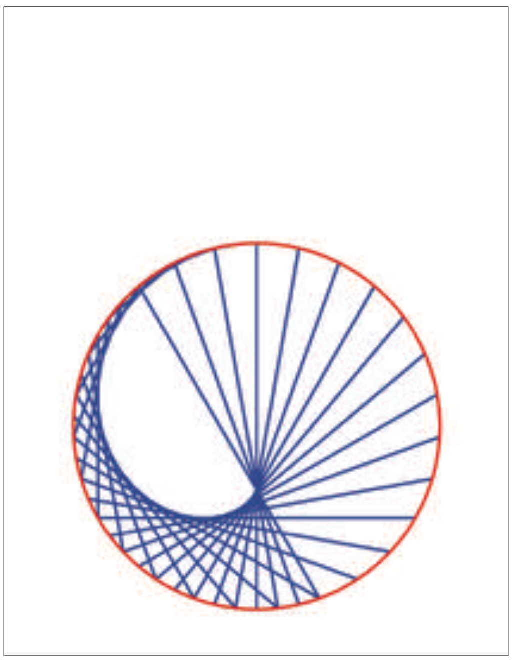
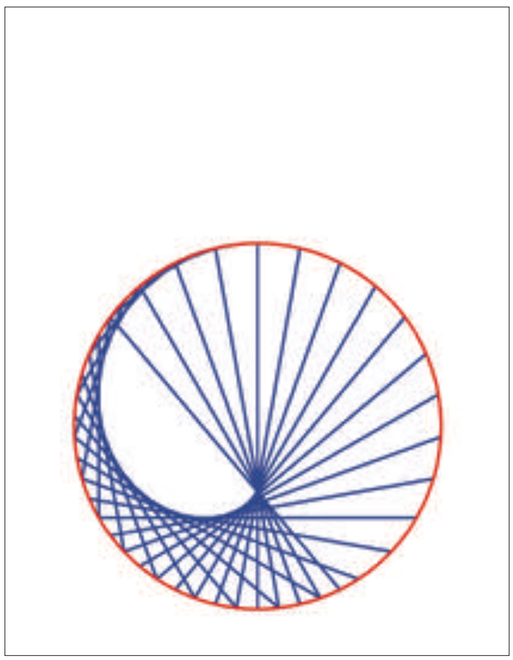
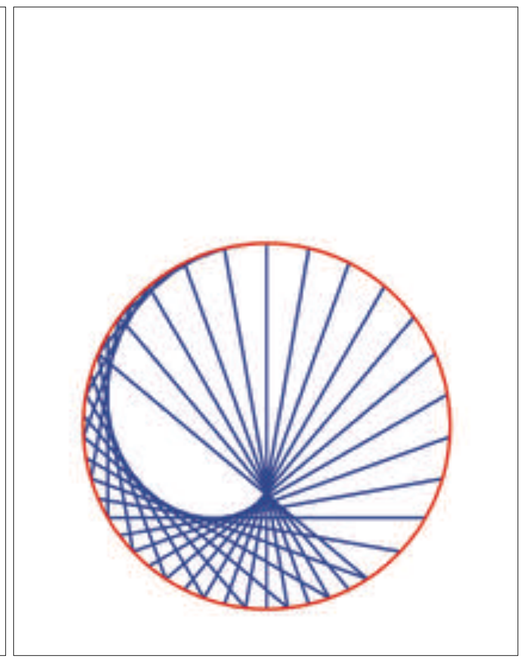
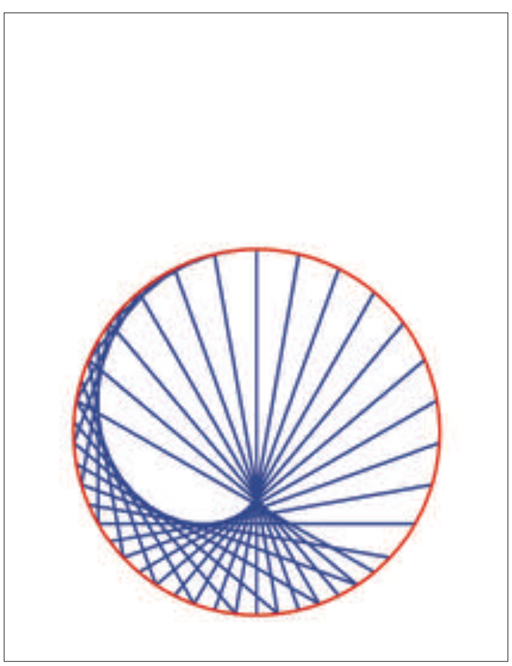
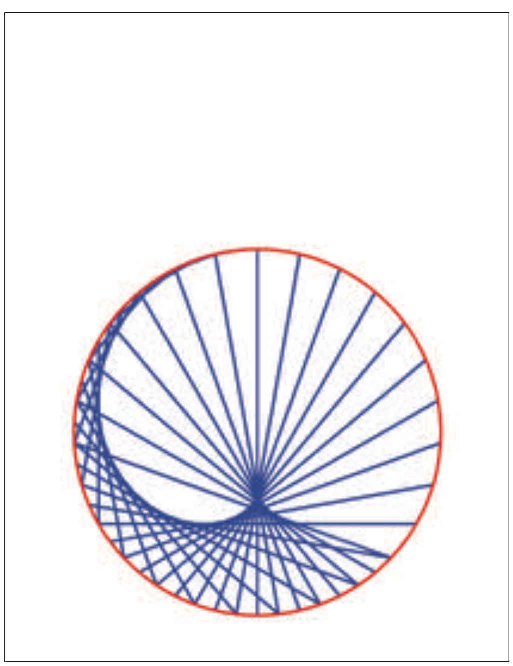
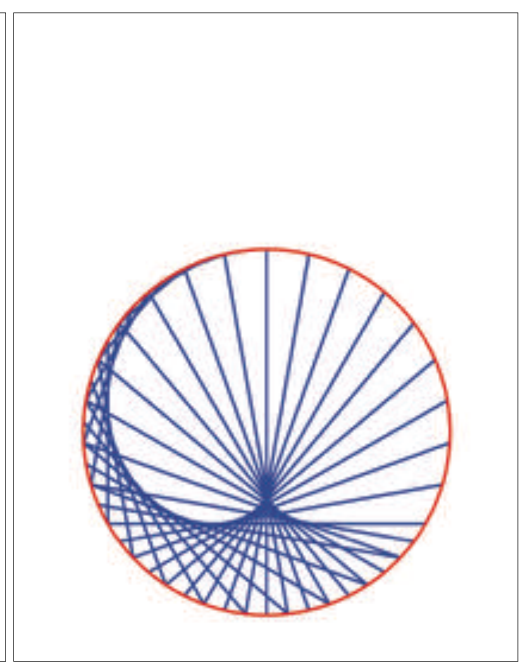

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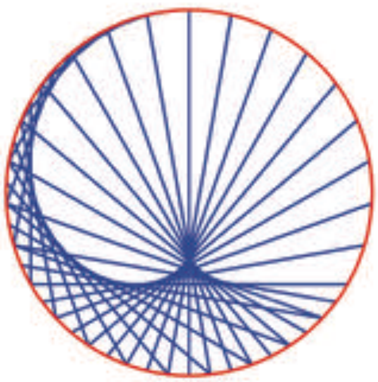
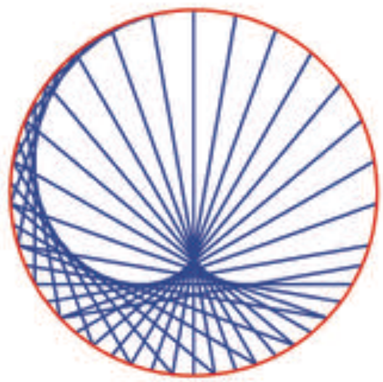
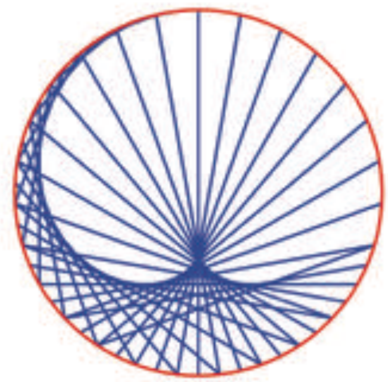
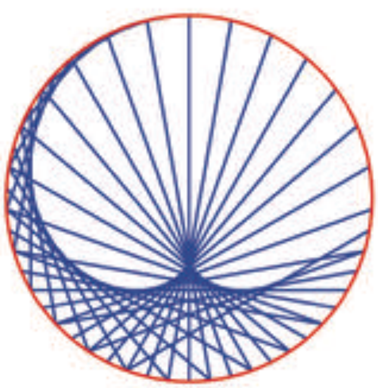
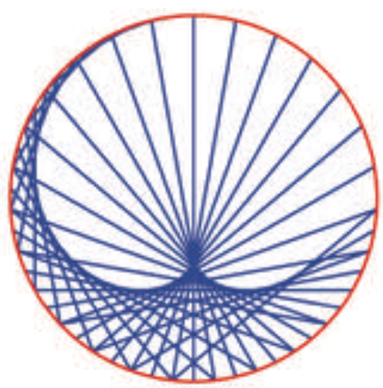
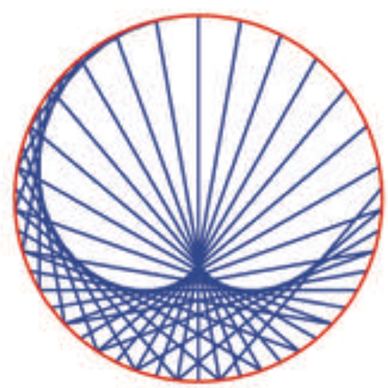
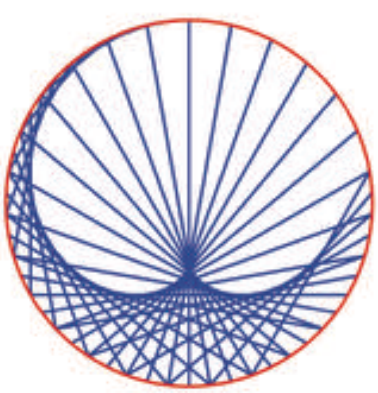
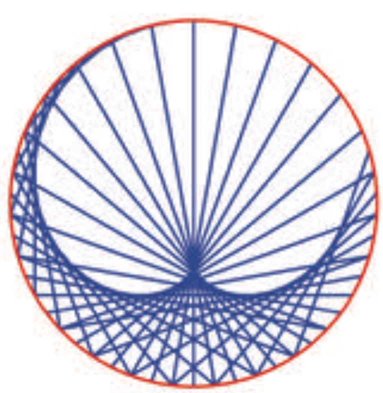
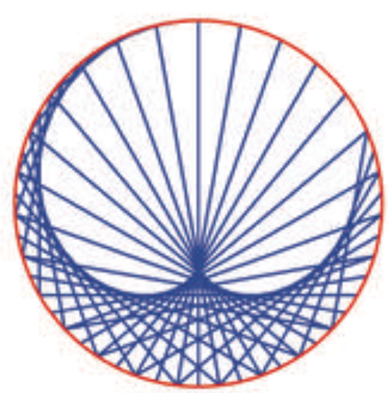

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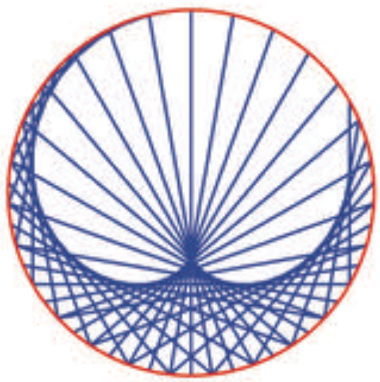
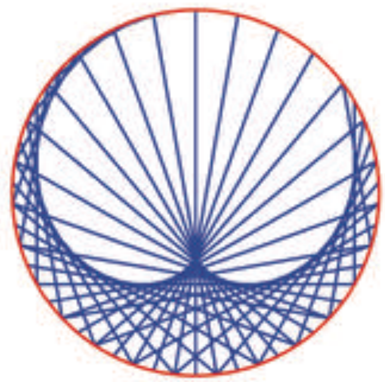
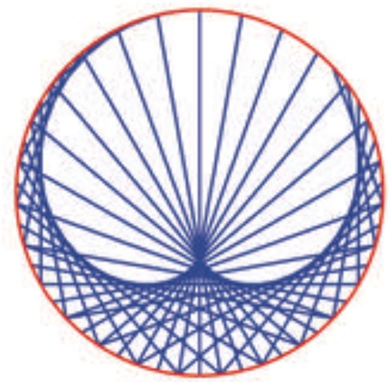
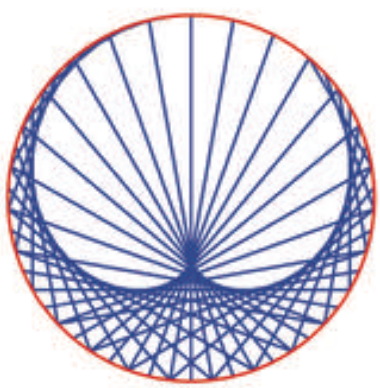
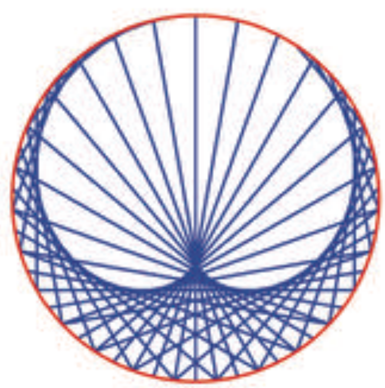
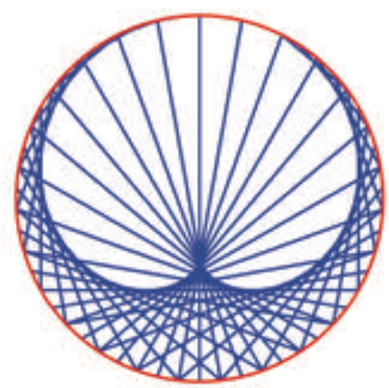
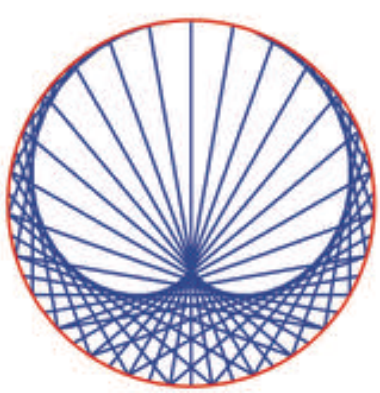
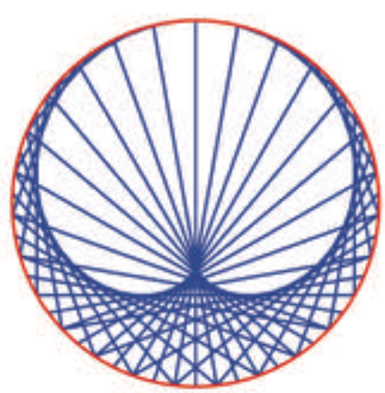
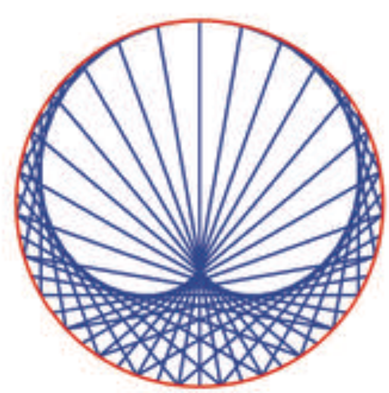

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53	50	47
52	49	46