A TWO-BUTTON CALCULATOR

This puzzle asks students to exercise the power of place-value and the effect of multiplying a number by the value of the base in which it is presented. (Multiplying a number written in base ten, for instance, by 10 has the apparent effect of appending a 0 to the number.)

EXPLODING DOTS Topic:

Experience 2: Understanding the place-value machines.
Experience 4: Multiplying by the base number in a machine.

Suggested Grade Level:

Middle-school and up.

GLOBAL MATH PROJECT

A TWO-BUTTON CALCULATOR

Present to your students the following three puzzles, either one at a time or altogether at the same time.

Puzzle 1: A calculator, currently displaying 0, has just two buttons.

$+$ Pressing this button adds 1 to the number currently on display.

$\times$ Pressing this button multiplies the number currently on display by ten.

By pressing these two buttons as many times as you wish, in any order you like, is it possible to make the number 5304 appear on screen?

If so, what is the least number of button presses needed to do it?

Puzzle 2: A calculator, currently displaying 0, has just two buttons.

$\times$ Pressing this button adds 1 to the number currently on display.

$+$ Pressing this button doubles the number currently on display.

By pressing these two buttons as many times as you wish, in any order you like, is it possible to make the number 244 appear on screen?

If so, what is the least number of button presses you needed to do it?

Puzzle 3: A calculator, currently displaying 0, has just two buttons.

$+$ Pressing this button adds 1 to the number currently on display.

$\times$ Pressing this button triples the number currently on display.

By pressing these two buttons as many times as you wish, in any order you like, is it possible to make the number 244 appear on screen?

What is the least number of button presses needed to do it?

Some Things Students Might Notice or Ask

1. You can always just repeatedly press the $+$ button (and never press the $\times$ button). That way you can get any number to appear on display. But it will take 5034 presses to make 5304 to appear in the first puzzle, and 244 presses in each of the remaining two puzzles to make 244 appear.

2. Pressing the $\times$ first never does you any good: multiplying zero by ten (or by two or by three) keeps you at zero.

3. You can get to 5000, at least, in the first puzzle by pressing $+$ five times and then $\times$ three times.

4. You can get to 5304 in the first puzzle by

> Pressing $+$ five times.
> Pressing $\times$.
> Pressing $+$ three times.
> Pressing $\times$.
> Pressing $\times$.
> Pressing $+$ four times.

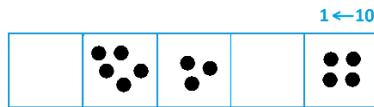That's fifteen button presses. Can we do it in less?

5. Puzzle 1 is hard enough!

6. In puzzle 1, would it ever be efficient to press the $+$ button ten times in a row?

SOLVING THE PUZZLE

> **See EXPERIENCES 2 and 4 of EXPLODING DOTS: Understanding the place-value machines.**

Let's focus on puzzle 1 for now.

Consider representing numbers in a $1 \leftarrow 10$ machine.

$1 \leftarrow 10$

•••••	••		•• ••

Now each press of the $+$ adds a dot to the rightmost box. Each press of the $\times$ button has the effect of shifting all the dots in the machine one place to the left.

$1 \leftarrow 10$
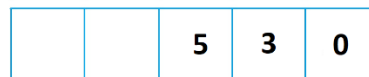
				0

Press $+$ five times.

				5

Press $\times$.

				50

			5	0

Press $+$ three times.

			5	3

Press $\times$.

		50	30

		5	3	0

Press $\times$.

	50	30	0

	5	3	0	0

Press $+$ four times.

5	3	0	4

We see we can reach the number 5304 in fifteen button presses.

We can reach any four-digit number $a \mid b \mid c \mid d$ by

> Pressing $+$, a times
> Pressing $\times$
> Pressing $+$, b times
> Pressing $\times$
> Pressing $+$, c times
> Pressing $\times$
> Pressing $+$, d times

for a total of $a + b + c + d + 3$ button presses, even if the "digits" a, b, c, d are larger than nine. For example, in Exploding Dots notation we can create $4 \mid 13 \mid 0 \mid 4$ in 24 button presses, which will display as 5304 on the calculator.

We could also create $4 \mid 12 \mid 10 \mid 4$ or $3 \mid 21 \mid 19 \mid 14$ or any other representation of 5304. The question is: *Would it ever behoove us to create more than ten dots in box by pressing the $+$ button more than ten times in a row?*

a	b		a-1	b+10

versus

Creating $a \mid b$ within part of a display requires pressing $+$, a times, pressing $\times$, pressing $+$, b times, for a total of $a + b + 1$ button presses.

Creating $a - 1 \mid b + 10$ within part of a display requires pressing $+$, $a - 1$ times, pressing $\times$, pressing $+$, $b + 10$ times, for a total of $a + b + 10$ button presses.

It is thus always more efficient to create the version of 5304 for which all explosions have occurred, that is, the version $5 \mid 3 \mid 0 \mid 4$ of the number for a total of 15 button presses. This number cannot be created in fewer button presses.

Puzzles 2 and 3:

Puzzle 2 is the same puzzle but based on base two: the $+$ adds a dot in the rightmost box of a $1 \leftarrow 2$ machine code and the $\times$ button shifts all the dots in the machine one place to the left.

The number 244 in base two is 11110100 and this can be obtained by pressing the buttons $+\times+\times+\times+\times\times+\times\times$ in order for a total of twelve button presses. This is optimal by essentially the same argument as before.

Puzzle 3 is the same puzzle but based on base three: the $+$ adds a dot in the rightmost box of a $1 \leftarrow 3$ machine code and the $\times$ button shifts all the dots in the machine one place to the left.

The number 244 in base three is 100001 and this can be obtained by pressing the buttons $+\times\times\times\times\times+$ in order for a total of seven button presses. This is optimal by essentially the same argument as before.

EXTENSIONS

Every solved problem, of course, is an invitation to explore and play more. Might your students enjoy these explorations?

Wild Exploration 1: A calculator, currently displaying 0, has just two buttons.

$+$ Pressing this button adds 1 to the number currently on display.

$\times$ Pressing this button multiplies the number currently on display by -2.

   a) Is it possible for each and every integer (positive or negative) to be on display?

   b) Analyze the minimum number of button presses needed to reach given integers.

Wild Exploration 1: A calculator, currently displaying 0, has just two buttons.

$+$ Pressing this button adds 1 to the number currently on display.

$\times$ Pressing this button multiplies the number currently on display by $\dfrac{3}{2}$ if the number is even, does nothing if the number is odd.

Which numbers are possible to display on this calculator?